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PROOF THEORY AND CATEGORY THEORY

1. Introduction

Our aim is to present some relations between proof theory and category theory which allow one to solve some algorithmic problems of category theory. Investigations of this kind were apparently initiated by Lambek [1] - [4]. We shall present his results concerning cartesian closed categories and their connections with the intuitionistic (&, $\supset$)-calculus using improvements from [5] and replacing derivations in a deductive system (which are described by graphs) by suitable terms which are linear objects. The novel features of our treatment are the translation of category-theoretic notions into the language of proof theory and the details of the proof of faithfulness of our translation of canonical maps into terms: if the translations of two maps are equal, then the maps themselves are equal. The author found detailed proofs neither in [4] nor in [5].

We use almost no material from category theory and almost all facts required from proof theory are presented with complete proofs. Nevertheless it is advisable for the reader to be acquainted at least superficially with elementary notions of category theory (categories, functors, natural transformations) and of proof theory (natural deduction systems, normalization of natural deductions).

Let us fix some notations. The = sign will stand for the usual (extensional) equality and $\underset{\cdot}{=}$ for (literal) coincidence of syntactic expressions.

A category $\mathfrak{K}$ consists of a class *Ob* $\mathfrak{K}$ of objects and a class Mor $\mathfrak{K}$ of maps (morphisms) satisfying the following conditions:

1) A set $H(A,B) \subset \mathrm{Mor}\ \mathfrak{K}$ is assigned to any ordered pair (A,B) of objects; $H(A,B)$ for different pairs (A, B) are disjoint and their union is Mor $\mathfrak{K}$; elements of $H(A,B)$ are called maps from A to B; one often writes $f: A \rightarrow B$ or $A \xrightarrow{f} B$ instead of $f \in H(A,B)$.

2) For any $f: A \rightarrow B$, $g: B \rightarrow C$, the composition $(g \cdot f): A \rightarrow C$ is defined; composition is associative, i.e. $(f \cdot (g \cdot h)) = (f \cdot g) \cdot h$; the outermost brackets and the point in the notation for composition are usually dropped.

3) For any $A \in Ob\, \mathfrak{R}$ there is the identity map $1 = 1_A : A \rightarrow A$ such that $1 \cdot f = f$, $g \cdot 1 = g$ for any $f\colon B \rightarrow A$, $g : A \rightarrow C$.

A detailed introduction to category theory with numerous examples is presented in [6].

Let us describe an example of a category which is basic in the sequel. *Objects* are formulas of some formal systems. *Maps* from A to B are derivations of B from the assumption A according to the rules of the system in question. The identity map 1_A is the trivial derivation of A from A consisting of A itself. Composition $g \cdot f$ of maps is the result of superimposing f over g.

2. Cartesian closed categories: The system *HCC*

In the following we shall use a scheme which can be traced to works of Lambek. To analyze the structure of maps in categories with a given kind of additional structure $\mathbb{S}$ one presents the free category of this kind in the form of a deductive system $H\, \mathbb{S}$ so that objects are formulas and maps are equivalence classes of derivations modulo a specific equivalence relation. This deductive system is reminiscent of those familiar from elementary textbooks in mathematical logic; they are sometimes called Hilbert-type systems, hence H in $H\, \mathbb{S}$. The equivalence relation on derivations is defined in such a way that equivalence classes obviously have the structure $\mathbb{S}$, but it is not obvious how to check directly (for given A,B) whether there is at least one $f\colon A \rightarrow B$ or (for given f, g) whether $f = g$ (that is whether the derivation f is equivalent to the derivation g). To solve these problems a natural deducation system, say $N\, \mathbb{S}$, is introduced and proved to be equivalent to $H\, \mathbb{S}$ not only with respect to derivable «sequents» $A \rightarrow B$ (as is usually done for various formulations of the same logical system) but also with respect to equality of derivations. A translation τ from $H\, \mathbb{S}$-derivations to $N\, \mathbb{S}$-derivations is defined in such a way that $\tau(f)$ for $f\colon A \rightarrow B$ is also a derivation of the sequent $A \rightarrow B$ (or a derivation of B from the single assumption A if one does not use sequent notation) and derivability in $N\, \mathbb{S}$ implies derivability in $H\, \mathbb{S}$. Moreover, $f = g$ iff $\tau(f)$ and $\tau(g)$ have the same normal form as defined for derivations. If $N\, \mathbb{S}$ is such that the derivability problem is solvable and any natural deduction (that is a derivation in $N\, \mathbb{S}$) has a unique normal form, then both of the problems above are solved. We shall carry out this scheme for cartesian closed categories.

A cartesian closed category $D = (D_0, I, \&, \supset, O, \langle\rangle, /, /, \varepsilon, {}^{+})$ is defined by the following data:

(i) category D_0;
(ii) object $I \in D_0$;
(iii) binary operations $\&$, $\supset$ from $Ob\, D_0$ into $Ob\, D_0$;

(iv) families of maps, indexed by objects, where the domains and codomains of these maps are formed from their indices in the obvious way.

$$O = O_A : A \to I;\ \ell = \ell_{AB} : A\&B \to A;\ \mathbf{r} = \mathbf{r}_{AB} : A\&B \to B \qquad (1)$$

$$\varepsilon = \varepsilon_{AB} : (A \to B)\&A \to B \qquad (2)$$

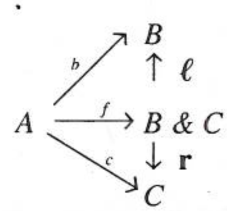
(v) a binary operation $\langle\rangle$ transforming any pair of maps $A \xrightarrow{b} B$, $A \xrightarrow{c} C$ into a map $\langle b,c\rangle : A \to B \,\&\, C$;

(vi) unary operation $^+$ transforming any map $A \,\&\, B \xrightarrow{c} C$ into a map $A \xrightarrow{c^+} B \supset C$.

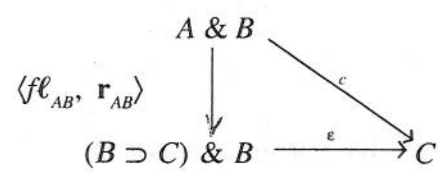
These must satisfy the following conditions.

D1. I is a terminal object, that is $O_A = f$ for all $f : A \to I$.

D2. For any $b : A \to B$, $c : A \to C$ the map $\langle b,c\rangle$ is the only map f such that the following diagram commutes:



D3. For any $c : A\&B \to C$ the map c^+ is the only map f such that the following diagram commutes:



Let us write these conditions in linear notation.

DA1. $b1_A = 1_B b = b$ for $b : A \to C$ (identity property)

DA2. $d(cb) = (dc)b$ for $A \xrightarrow{b} B \xrightarrow{c} C \xrightarrow{d} D$ (associativity of composition).

DA3. $O_A = f$ for $f : A \to I$ (terminal object property for I).

DA4. $\ell_{BC}\,\langle b,c\rangle = b$; $\mathbf{r}_{BC}\,\langle b,c\rangle = c$ for $A \xrightarrow{b} B$, $A \xrightarrow{c} C$ (commutativity of D2 for $f = \langle b,c\rangle$).

DA5. $\langle \ell f, \mathbf{r}f\rangle = f$ for any $f : A \to B\&C$ (uniqueness of f in D2).

DA6. $\varepsilon\langle c^{+}\ell_{AB}, \mathbf{r}_{AB}\rangle = c$ for all $c : A\&B \to C$ (commutativity of D3).

DA7. $\left(\varepsilon\langle f\ell_{AB}, \mathbf{r}_{AB}\rangle\right)^{+} = f$ for all $f : A \to B \supset C$ (uniqueness of f in D3).

In the following we shall use only these conditions and the properties of equality.

We shall be interested in canonical maps in cartesian closed categories, that is maps which can be obtained from 1, O, ℓ, $\mathbf{r}$, ε by means of $\langle\rangle$, $^{+}$ and composition.

The system HCC (the Hilbert-type system for cartesian closed categories).

Atomic formulas are propositional variables and the constant I. Formulas are constructed from the atomic ones by means of the binary connectives &, $\supset$. The sequents of the system *HCC* are expressions of the form $A \to B$, where A,B are formulas.

Derivations and derivable sequents are defined inductively. The notation $A \overset{b}{\to} B$ or $b : A \to B$ means «b is a derivation of $A \to B$». Let us list (1.1 - 2.3) the axioms and inferences rules.

1. The category postulates (*)

1.1. $1_A : A \to A$ (identity).

1.2. If $b : A \to B$, $c : B \to C$ then $cb : A \to C$ (composition).

This derivation is displayed as follows in tree form:

$$\frac{A \overset{b}{\to} B; \qquad B \overset{c}{\to} C}{A \overset{(cb)}{\to} C}$$

Brackets are usually dropped in the notation for composition.

2. Postulates for cartesian closed categories

2.1. Axioms (1), (2) above introducing O, ℓ, $\mathbf{r}$, ε.

B&C

2.2. $$\frac{A \overset{b}{\to} B; \qquad A \overset{c}{\to} C}{A \overset{\langle b,c\rangle}{\to} C}$$

2.3. $$\frac{A \,\&\, B \overset{c}{\to} C}{A \overset{c^{+}}{\to} B \supset C}$$

(*) Postulates include both the axioms and the inference rules.

Derivations in *HCC* will also be called *combinators* to indicate the relation to combinatory logic [8].

We define a *relation* $\equiv$ for combinators as the least congruence turning *HCC* into a cartesina closed category. In a more detailed way the relation $\equiv$ is defined inductively by axioms DA1-DA7 (with = replaced by $\equiv$), the axiom $a \equiv a$ and the rules:

$$\frac{a \equiv b \quad a \equiv c}{b \equiv c} \qquad \frac{a \equiv b \quad c \equiv d}{\langle a,c \rangle \equiv \langle b,d \rangle} \qquad \frac{a \equiv b \quad c \equiv d}{(ac) \equiv (bd)}$$

$$\frac{a \equiv b}{a^{+} \equiv b^{+}} \qquad \frac{a \equiv b}{\ell a \equiv \ell b} \qquad \frac{a \equiv b}{\mathrm{r}a \equiv \mathrm{r}b}$$

A sequent $A \to B$ is said to be *valid* (for ~~certain~~ cartesian closed categories) if for any cartesian closed category $\mathfrak{R}$ and any substitution ζ of objects from $\mathfrak{R}$ for propositional variables in $A \to B$, one has a map $b : \zeta A \to \zeta B$ in Mor $\mathfrak{R}$. Two canonical maps are considered *equal* if their realizations are equal in any cartesian closed category. We shall consider the problem of recognizing the validity of sequents (the word problem) and that of recognizing equality of canonical maps (the coherence problem). The following simple observation shows that it is sufficient to solve these problems for *HCC*.

LEMMA 2.1. $A \to B$ is derivable in *HCC* if and only if it is valid. Canonical maps a,b are equal if and only if $a \equiv b$.

Proof. If $A \xrightarrow{b} B$ in *HCC* then for any cartesian closed category $\mathfrak{R}$ and for any substitution ξ we have $b_\xi : \xi A \to \xi B$, where b_ξ is the result of applying ξ to all indices of maps in b. Conversely, validity of $A \to B$ implies in particular that there is $b : A \to B$ in *HCC*. The second part of the lemma is proved similarly.

Note that in view of the lemma it is impossible that $A \to B$ be valid but realizable only by different maps in different categories $\mathfrak{R}$: there is always a unique map (one for *HCC*) which works for all $\mathfrak{R}$.

By &-maps we shall mean combinators constructed without any use of ε and $^{+}$. For the same pair A,B of formulas there may exist different &-maps $A \to B$, for example

$$\langle \mathrm{r},\ell \rangle : A\&A \to A\&A; \qquad 1: A\&A \to A\&A$$

This situation is however impossible if there is no identification of variables in A,B. We shall now establish this using some ideas from [7] with the considerable simplifications possible in this situation.

THEOREM 2.2. Let A, B be conjunctions constructed from I and variables such that no variable occurs twice in A and each variable from B is in A. Then there exists a map

$$\alpha_{AB} : A \to B$$

which is unique (modulo $\equiv$). In particular, if the same variables occur in A and B and no variable occurs twice in either of A, B, then α_{AB} is an isomorphism:

$$\alpha_{AB}\, \alpha_{BA} \equiv 1.$$

Proof. The basis of all the following arguments is «the passage to (left and right) projections»:

$$a \equiv b \quad \text{if and only if } \ell a \equiv \ell b \ \text{ and } \mathrm{r}a \equiv \mathrm{r}b \tag{3}$$

which follows from DA5: $a \equiv \langle \ell a, \mathrm{r}a \rangle \equiv \langle \ell b, \mathrm{r}b \rangle \equiv b$.

In particular (3) implies distributivity of composition over pairing.

$$\langle a,b\rangle\, c \equiv \langle ac, bc\rangle \tag{*}$$

since $\ell(\langle a,b\rangle\, c) \overset{\text{assoc.}}{\equiv} (\ell\langle a,b\rangle)\, c \equiv ac$ and $\mathrm{r}(\langle ab\rangle\, c) \equiv bc$, and hence (*) by DA5.

Let us prove the theorem by induction on the length of the formula B. The induction step ($B \doteq B_1 \,\&\, B_2$) is obvious: having α_{AB_1} and α_{AB_2} we put $\alpha_{AB} \doteq \langle \alpha_{AB_1}, \alpha_{AB_2}\rangle$. If we have $b : A \to B_1 \,\&\, B_2$ then $\ell b\colon A \to B_1$, $\mathrm{r}b : A \to B_2$ so that $\mathrm{l}b \equiv \alpha_{AB_1}$, $\mathrm{r}B \equiv \alpha_{AB_2}$ by the induction hypothesis and $b \equiv \alpha_{AB}$ by (3).

We are left with the induction base when B is a variable or I. The case $B \doteq I$ is trivial because I is the terminal object. If B is a variable, we apply induction on the length of A to construct α_{AB} as a chain of ℓ,r.

If A is a variable (other than B) then $A \doteq B$ would not even be a two-valued tautology, hence it would be underivable in *HCC*. Put $\alpha_{BB} \doteq 1$. If $A \doteq A_l \& A_r$ we put $\alpha_{AB} = \alpha_{A_l B}\ell$ or $\alpha_{AB} = \alpha_{A_r B}\,\mathrm{r}$ depending on where B occurs. If for example $A \doteq (C\&(B\&D))\&E$ then $\alpha_{AB} \doteq 1\ell\mathrm{r}\ell \equiv \ell\mathrm{r}\ell$.

To prove uniqueness (when B is a variable) we employ a method which later will allow us to solve the general coherence problem. Let us note that for any &-map a there is an &-map a' such that $a \equiv a'$ and, modulo the associativity of composition, a' contains no part of the form

$$\ell\langle c,d\rangle;\ \mathrm{r}\langle c,d\rangle;\ 1c;\ c1 \tag{4}$$

In fact by DA1, DA2, DA4 one can regroup compositions and delete redundant parts, replacing for example $\ell\langle c,d\rangle$ and $1\bullet c$ by c.

Assume now that $b \equiv \alpha_{AB}$ and b contains (modulo associativity) no parts of the form (4). Using associativity transform b into an equivalent form $(\ldots (b_1\ b_2)\ \ldots\ b_n)$ with no b_i of the form cd. If $b_n = \langle c,d\rangle$ then B would not be a variable because there are no parts of the form (4). So either $n = 1$ and $b \mp 1 \mp \alpha_{AB}$ or $b_n = \ell$ or $b_n = \mathbf{r}$ depending on where the variable B occurs in the conjunction $A \mp A_l \& A_r$. Let us assume $b_n = \ell$. Then $b_0 \ldots b_{n-1}$: $A_l \to B$ so by induction hypothesis $b_0 \ldots b_{n-1} \equiv \alpha_{A_lB}$ hence $b \equiv \alpha_{A_lB}\, \ell \equiv \alpha_{AB}$ by the definition of α. The theorem is proved.

3. The system of deductive terms

To investigate the system *HCC* and the relation $\equiv$ we shall use a system of terms to be described below. Any term will have some formula A as its type and the terms of type A can be understood as solutions of problems generated by the formula A. Notation $t \in A$ stands for «t is a term of type A». Sometimes instead of writing $t \in A$ we shall add the superscript A to t.

(1) For any formula A there is a list of type A variables denoted by x^A, y^A, ... (possibly with subscripts).

(2) An expression (b,a) is a term of type B for any terms b,a of the type $A \supset B$, A respectively. This we shorten to $(b^{A\supset B}, a^A) \in B$.

(3) $\langle a^A,b^B\rangle \in \langle A\&B\rangle$.

(4) $\lambda x^A b^B \in (A \supset B)$.

(5) $\ell c^{A\&B} \in A$; $\mathbf{r}c^{A\&B} \in B$.

(6) $I \in I$.

Occurrences of variables x^A in terms are classified in the usual way into bound (by λxA) and free.

Comments (which are not used below but facilitate remembering notation and using it). Recall the constructive interpretation of deductive terms dating back to Kolmogorov and Heyting.

(1) x^A is a variable for solutions of the problems generated by the formula A [A-problems for short].

(2) An $(A \supset B)$-problem consists in producing a general method allowing one to construct a solution of the B-problem for any given solution of the A-problem. The expression (c,a) is interpreted as the result of the application of (function) c to a.

(3) An $A\&B$-problem consists in finding a pair $\langle a,b\rangle$ having a solution of the A-problem as its first component and a solution of the B-problem as its second component.

(4) If a solution $b^B[x^A]$ of a B-problem possibly depending on a parameter x^A is known, then a solution of the $(A \supset B)$-problem can be obtained in the form of a function $\varphi = \lambda x^A b^B$ such that $\varphi(a) = b^B[a]$ for all a.

Here $b_{x_1,\ldots,x_k}[c_1, \ldots, c_k]$ means the result of substituting $c_1, \ldots, c_k$ for all free occurrences of $x_1, \ldots, x_k$ respectively in b, renaming bound variables to avoid collision of variables. If $b \doteqdot \lambda y(y,x)$ and $c \doteqdot (y,x)$ then $b_x[c] \doteqdot \lambda z(z,(y,x))$. We shall often omit the subscripts $x_1, \ldots, x_n$.

(5) If a solution $x^{A\&B}$ of $(A\&B)$-problem is known then a solution of the A-problem (respectively the B-problem) is obtained by taking the left component or projection $\mathbf{l}c$ (right component $\mathbf{r}c$ respectively).

(6) I is interpreted as the trivial decidable problem (identical truth) and $\mathbf{I}$ as its trivial solution.

This system of deductive terms was introduced [8] - [11] in connection with the natural deduction calculus for intuitionistic logic ([12], [13]). Derivable objects of this calculus are sequents $\Gamma \to B$ where Γ is a list of formulas (possibly empty) and B is a formula. Axioms are sequents $A \to A$ and $\to I$ and terms x^A and $\mathbf{I}$ are assigned to these axioms respectively. We shall use for terms the same notation as for maps: $A \xrightarrow{x^A} A$, $\xrightarrow{\mathbf{I}} I$, $x^A : A \to A$, $\mathbf{I}\colon \to I$. The inference rules are as follows (modulo the order of antecedent formulas, i.e. ones to the left of the arrow):

$$(\supset^-)\ \frac{\Gamma\ \Pi \xrightarrow{b} (A \supset B);\ \Sigma\Pi \xrightarrow{a} A}{\Gamma\ \Sigma\ \Pi \xrightarrow{(b,\,a)} B} \qquad \frac{\Gamma\ \Pi \xrightarrow{a} A\ ;\ \ \Sigma\Pi \xrightarrow{b} B}{\Gamma\ \Sigma\ \Pi \xrightarrow{\langle a,b\rangle} (A\&B)}\ (\&^+)$$

$$(\supset^+)\ \frac{\Gamma\ A^0 \xrightarrow{b} B}{\Gamma \xrightarrow{\lambda x^A b} A \supset B)} \qquad \frac{\Gamma \xrightarrow{c} (A\&B)}{\Gamma \xrightarrow{\mathbf{l}c} A}\ (\&^-)\ \frac{\Gamma \xrightarrow{c} A\&B}{\Gamma \xrightarrow{\mathbf{r}c} A}$$

where the notation A^0 in the $\supset^+$-rule indicates that A may be absent from the upper sequent (the premiss of the rule). The list Π in the two-premiss rules consists of formula (occurrence)s to which identical free variables are assigned in terms a and b. The term $\langle\langle x^A,x^A\rangle, \langle x^A,y^A\rangle\rangle$ for example corresponds to the derivation

$$\frac{\&^+\ \dfrac{A \xrightarrow{x^A} A;\ A \xrightarrow{x^A} A}{A \to A\ \&\ A;} \quad \dfrac{A \xrightarrow{x^A} A;\ A \xrightarrow{y^A} A}{A,\ A \to A\&A}\ \&^+}{A,A \to ((A\&A)\ \&\ (A\ \&\ A))}\ \&^+$$

and Π in the lowermost $\&^+$ is A. So there is an isomorphism between terms and natural deductions and we shall use the latter only for illustration (*).

Consider for example the familiar inference

$$(cut)\quad \frac{\Pi \xrightarrow{a} A; \quad \Gamma\, A \xrightarrow{b} B}{\Gamma\Pi \xrightarrow{b_{x}\, A[a]} B}$$

which is easily simulated by the sequence

$$\frac{\dfrac{\Gamma\, A \xrightarrow{b} B}{\Gamma \xrightarrow{\lambda x^A b} A \supset B}; \quad \Pi \xrightarrow{a} A}{\Gamma\Pi \xrightarrow{(\lambda x^A b,\, a)} B}$$

Under the usual rules for the λ-symbol to be described below, the term corresponding to the latter deduction is transformed into $b_{x}\,{}_{A[a]}$.

Now we shall assign a term $\tau(b)$ of type B to any canonical map $b : A \to B$. No free variable except possibly x^A will occur in $\tau(b)$.

$\tau(1_A) \doteq x^A;\quad \tau(O_A) \doteq \mathbf{I};$

$\tau(\ell_{AB}) \doteq \ell x^{A\&B};\qquad \tau(\mathbf{r}_{AB}) \doteq \mathbf{r}x^{A\&B};$

$\tau(\langle b,c\rangle) \doteq \langle\tau(b),\ \tau(c)\rangle$ for $A \xrightarrow{b} B,\ A \xrightarrow{c} C;$

$\tau(\varepsilon_{AB}) \doteq (\ell x,\ \mathbf{r}x)$ where $x \doteq x^{(A\,\supset\, B)\&A};$

$\tau(c^+) \doteq \lambda x^B(\lambda x^{A\&B}\ \tau(c),\ \langle x^A,\ x^B\rangle)$ for $c : A\&B \to C;$

$\tau((b\cdot c)) \doteq (\lambda x^C\ \tau(b))$ for $A \xrightarrow{c} C \xrightarrow{b} B.$

To define the inverse translation of terms into combinators we shall contract lists $\Gamma \doteq A_1, \ldots, A_n$ into conjunctions $K(\Gamma) \doteq (\ldots(I\&A_1)\ \&\ \ldots\ \&\ A_n)$ which for $n = 0$ is simply I. To treat interrelations between lists Γ,Δ we would like to define $\&$-maps $\alpha_{\Gamma,\Delta} : K(\Gamma) \to K(\Delta)$. Different occurrences of the same formula as a member of such a list are to be treated here as if they were occurrences of different formulas. To facilitate this we use the fact that these different occurrences of the same formula A in a sequent $\Pi A \Sigma A \to D$ are assigned different free variables x^A, y^A in any term t: $\Pi A\ \Sigma\ A \to D$. So

(*) For readers accustomed to the definition of a derivation as a formula (not sequent) tree we note that our derivation of a sequent $\Gamma \to A$ corresponds to the derivation of the formula A from undischarged assumptions Γ.

in fact we shall define $\alpha_{\Gamma,\Delta}$ for non-empty lists of variables $x_i^{A_i}$ such that all variables in Γ are distinct and all variables from Δ occur in Γ.

Let $\Gamma \doteqdot x_1^{A_1} \dots x_n^{A_n}$ and $\Delta = x_{n+1}^{A_{n+1}} \dots x_m^{A_m}$. Choose n distinct propositional variables $p_1, \dots, p_n$ and introduce the notation:

$$K'(\Gamma) \doteqdot (\dots (p_1 \,\&\, p_2) \,\&\, \dots \,\&\, p_n);$$

$$K'(\Delta) \doteqdot (\dots (p'_{n+1} \,\&\, p'_{n+2}) \,\&\, \dots \,\&\, p'_m);$$

$$K(\Gamma) \doteqdot (\dots (I \,\&\, A_1) \,\&\, \dots \,\&\, A_n),$$

where $p'_j \doteqdot p_{i_j}$ $(n+1 \le j \le m;\ 1 \le i_j \le n)$ iff $x_j^{Aj} = x_{i}^{Ai}$. By theorem 2.2. there exists a unique map $\alpha'_{\Gamma,\Delta} \doteqdot \alpha_{K'(\Gamma),K'(\Delta)} : K'(\Gamma) \to K'(\Delta)$. Put

$$\alpha_{\Gamma,\Delta} \doteqdot \alpha'_{\Gamma,\Delta}\,[p_i/A_i]$$

where the right hand side stands for the result of replacing indices p_i by A_i in the canonical maps constituting $\alpha'_{\Gamma,\Delta}$.

Let $\Gamma \sim \Gamma$ mean that Γ' is a permutation of Γ. By theorem 2.2. we have if $\Gamma \sim \Gamma'$ then $\alpha_{\Gamma,\Gamma'}$ is an isomorphism:

$$\alpha_{\Gamma',\Gamma}\ \alpha_{\Gamma,\Gamma'} \equiv 1. \tag{2}$$

$$\text{If } \Gamma \sim \Gamma' \text{ then } \alpha_{\Gamma',\Sigma} \equiv \alpha_{\Gamma,\Sigma}\ \alpha_{\Gamma',\Gamma};\ \equiv \alpha_{\Sigma,\Gamma'}\ \alpha_{\Sigma,\Gamma} \tag{3}$$

$$\alpha_{\Gamma,\Gamma} \equiv 1 \tag{4}$$

$$\alpha_{\Gamma x,\Delta x} \equiv \langle \alpha_{\Gamma,\Delta}\ell,\ r\rangle \text{ if } x \text{ is not in } \Gamma\Delta \tag{5}$$

$$\alpha_{\Gamma\Sigma,\Delta} \equiv \alpha_{\Gamma,\Delta}\ell^{\Sigma} \tag{6}$$

$$\ell^{\Sigma}\alpha_{\Gamma,\Delta\Sigma} \equiv \alpha_{\Gamma,\Delta} \tag{7}$$

where $\ell^{A_1 \dots A_n}$ stand for ℓ repeated n times with suitable indices.

For any term t^A and any list Γ of distinct variables containing all free variables from t^A (and possibly some others) we define by recursion on (the construction of) t^A a map

$$\mathbf{c}_\Gamma(t^A) : K(\Gamma) \to A.$$

Inclusion in Γ of some variables not in t^A is needed to «equalize» lists of free variables for terms a,b in clauses 2, 3 below

1. $\mathbf{c}_{\Pi x^A \Sigma}(x^A) \doteqdot \alpha_{x^J \Pi x^A \Sigma, x^A} \equiv \mathbf{r}\ell^{\Sigma}$.
2. $\mathbf{c}_\Gamma((b,a)) \doteqdot \varepsilon\ \langle \mathbf{c}_\Gamma(b),\ \mathbf{c}_\Gamma(a)\rangle$.

3. $\mathbf{c}_\Gamma(\langle a,b\rangle) \doteq \langle \mathbf{c}_\Gamma(a),\ \mathbf{c}_\Gamma(b)\rangle$.

4. $\mathbf{c}_\Gamma(\lambda x^A b) \doteq (\mathbf{c}_{\Gamma x'^A}(b'))^+$, where $x'^A \doteq x^A$ and $b' \doteq b$ if x^A does not occur in Γ; $x'^A \doteq y^A$ and $b \doteq b_{x^A}[y^A]$ if x^A occurs in Γ; y^A is a variable.

5. $\mathbf{c}_\Gamma(\mathbf{f}t) \doteq \mathbf{f}\mathbf{c}_\Gamma(t)$ for $f \doteq l,r$.

6. $\mathbf{c}_\Gamma(\mathbf{I}) \doteq O_{K(\Gamma)}$.

LEMMA 3.1. The sequent $A \to B$ is derivable in *HCC* iff there exists a term t^B containing free no variables except x^A.

Proof. If $A \xrightarrow{b} B$ in *HCC* then $\tau(b)$ is a required term. If t^B contains free no variable except x^A then $\mathbf{c}_{x^A}(t^B)$ is defined and $\mathbf{c}_{x^A}(t^B) : I\&A \to B$, hence $\mathbf{c}_{x^A}(t^B)\ \langle O_A,\ 1_A\rangle : A \to B$ as required.

4. The equivalence of terms. The relation between the translations τ and c

Our next task is to show that the translations τ and $\mathbf{c}$ are inverse modulo the multiplier $\langle O,\ 1\rangle$ and the equivalence relation $\equiv$, which was defined for maps (combinators) and now will be defined for terms as the least congruence satisfying relations (1) - (6) below.

$$c \equiv \lambda x^A(c,x^A) \text{ where } c \in (A \supset B) \text{ and } x^A \text{ is a new variable} \quad (1)$$

$$c \equiv \langle \ell c,\ \mathbf{r}c\rangle \text{ where } c \in (A\&B) \quad (2)$$

$t \equiv t'$ if t,t' are congruent, that is can be obtained from each other by renaming bound variables (3)

$$t^I \equiv \mathbf{I} \quad (4)$$

$$(\lambda x^A b, a^A) \equiv b_{x^A}[a^A] \quad (5)$$

$$\ell\langle a,b\rangle \equiv a; \qquad \mathbf{r}\langle a,b\rangle \equiv b \quad (6)$$

$$\frac{a \equiv b;\ a \equiv c}{b \equiv c} \qquad \frac{a \equiv b}{c[x/a] \equiv c[x/b]} \quad (7)$$

where $c[x/a]$ means the result of substituting a for all free occurrences of x in c without renaming of variables. So $a \equiv b$ implies $\lambda ya \equiv \lambda yb$. A demonstration of $a \equiv b$ is a derivation of $a \equiv b$ from axioms (1) - (6) by rules (7).

Let us prove several relations for combinators.

LEMMA 4.1. (i) $\langle a,b\rangle\ h \equiv \langle ah,\ bh\rangle$;

(ii) $\varepsilon\langle c^+,b\rangle \equiv c\langle 1,b\rangle$ for $A\&B \xrightarrow{c} C,\ A \xrightarrow{b} B$

(iii) $c^+a \equiv (c\langle a\ell,\mathbf{r}\rangle)^+$ for $A\&B \xrightarrow{c} C,\ A_1 \xrightarrow{a} A$

Proof. (i) distributivity is proved by the passage to projections, see (3) of section 2.

(ii) $\mathbf{c}\langle 1,\mathbf{b}\rangle \underset{DA6}{\equiv} \varepsilon\langle c^+\ell,\mathbf{r}\rangle\ \langle 1,b\rangle, \underset{(i)}{\equiv} \varepsilon\ \langle c^+\ell\ \langle 1,b\rangle,\ \mathbf{r}\ \langle 1,b\rangle\rangle \underset{DA4,\ DA1}{\equiv} \varepsilon\ \langle c^+,\ b\rangle$

(iii) $c^+\ a \underset{DA7}{\equiv} (\varepsilon\langle c^+a\ell,\mathbf{r}\rangle)^+ \underset{(i)DA4}{\equiv} (\varepsilon\langle c^+\ell,\mathbf{r}\rangle\ \ \langle a\ell,\mathbf{r}\rangle)^+ \underset{DA6}{\equiv} (c\langle a\ell,\mathbf{r}\rangle)^+$.

The lemma is proved.

For combinators $b : A \to B$ we have $\mathbf{c}_{x^A}(\tau(b)) : I\&\ A \to B$.

LEMMA 4.2. $\mathbf{c}_{x^A}(\tau(b)) \equiv b\mathbf{r}$ for $b : A \to B$.

Let us put $\tilde{b} \doteqdot \mathbf{c}_{x^A}(\tau(b))$. Then our lemma is

$$\tilde{b} \equiv b\mathbf{r}.$$

We prove it by induction on b. Indices of $\mathbf{c}$ are dropped if they do not change during the computation in question.

1. $\tilde{O} \doteqdot \mathbf{c}(I) \doteqdot O,\ O\mathbf{r} \equiv O$ by DA3.

2. For $f \doteqdot l,r$ we have $\tilde{\mathbf{f}} \doteqdot \mathbf{c}(\mathbf{f}x^{A\&B}) = \mathbf{f}\mathbf{c}(x^{A\&B}) \doteqdot \mathbf{f}\alpha_{x^I x^{A\&B},x^{A\&B}} \equiv \mathbf{fr}$.

3. $\langle\tilde{b},c\rangle \doteqdot \mathbf{c}(\langle\tau(b),\ \tau(c)\rangle) \doteqdot \langle\tilde{b},\tilde{c}\rangle \equiv \langle b\mathbf{r},\ c\mathbf{r}\rangle \equiv \langle b,c\rangle\mathbf{r}$.

The first $\equiv$ holds by the induction hypothesis and the second one by distributivity.

4. $\tilde{\varepsilon} \doteqdot \mathbf{c}((\ell x,\ \mathbf{r}x)) \doteqdot \varepsilon\ \langle\mathbf{c}(\ell x),\ \mathbf{c}(\mathbf{r}x)\rangle \equiv \varepsilon\langle\ell\mathbf{r},\mathbf{rr}\rangle \equiv \varepsilon\langle\ell,\mathbf{r}\rangle\ \mathbf{r} \equiv \varepsilon 1\mathbf{r} \equiv \varepsilon\mathbf{r}$ by distributivity and DA5.

$$5.\ \tilde{d}^+ \doteqdot \mathbf{c}_{x^A}(\lambda x^B\ (\lambda x^{A\&B}\ \tau(d),\ \langle x^A,x^B\rangle))) \doteqdot \mathbf{c}_{x^A x^B}^{+}\ (\lambda x^{A\&B}\ \tau(d), \langle x^A,x^B\rangle)) \doteqdot$$
$$(\varepsilon\langle\mathbf{c}^+_{x^A x^B x^{A\&B}}(\tau(d)),\ \langle\mathbf{c}_{x^A x^B}\ (x^A),\ \mathbf{c}_{x^A x^B}\ (x^B)\rangle\rangle)^+ \equiv (\varepsilon\langle\tilde{d}^+;\ \langle\mathbf{r}\ell\mathbf{r},\mathbf{r}\rangle)^+$$

where d: $A\&B \to C,\ d^+ : A \to B \supset C$ and $x^B,\ x^{A\&B}$ are assumed not to occur in Γ to simplify notation. We proceed by using the induction hypothesis and lemma 4 (ii);

$$\varepsilon\langle\tilde{d}^+,\ \langle\mathbf{r}\ell,\mathbf{r}\rangle\rangle \equiv \varepsilon\langle(d\mathbf{r})^+,\ \langle\mathbf{r}\ell,\mathbf{r}\rangle\rangle \equiv d\mathbf{r}\langle 1,\mathbf{r}\ell,\mathbf{r}\rangle\rangle \underset{DA4}{\equiv} d\langle\mathbf{r}\ell,\mathbf{r}\rangle$$

Hence $\tilde{d}^{+} \equiv (d\langle \mathbf{r}\ell, r\rangle)^{+}$. On the other hand

$$d^{+}\ \mathbf{r} \equiv (\varepsilon\langle d^{+}\ \mathbf{r}\ell,\ \mathbf{r}\rangle)^{+} \underset{\text{(i)DA4}}{\equiv} (\varepsilon\langle d^{+}\ \ell,\ \mathbf{r}\rangle)\ \ \langle \mathbf{r}\ell,\ \mathbf{r}\rangle)^{+} \underset{\text{DA6}}{\equiv} (d\ \langle \mathbf{r}\ell,\ \mathbf{r}\rangle)^{+}.$$

6. $(\tilde{b}\cdot c) \doteqdot \mathbf{c}_{x^A}(\lambda x^c \tau(b),\ \tau(c)) \doteqdot \varepsilon\langle \mathbf{c}+_{x^A,x^c}\tau(b),\ \mathbf{c}_{x^A}\tau(c)\rangle$

$$\underset{\text{(ii)}}{\equiv} b\mathbf{r}\langle 1,\ c\mathbf{r}\rangle \equiv (b\cdot c)\mathbf{r}$$

LEMMA 4.2. is proved.

LEMMA 4.3. For any combinators a,b

$a \equiv b$ implies $\tau(a) \equiv \tau(b)$

if the free variables for the construction of $\tau(a)$, $\tau(b)$ are chosen in the same way.

The proof is by induction on the demonstration of $a \equiv b$. Let us check axioms DAi. The notation $\underset{K}{\equiv}$ indicates that the equivalence is by formula (k), $k = 1, \ldots, 6$ in this section.

DA1. $\tau((b\ 1)) \doteqdot (\lambda x\tau(b),\ \tau(1)) \doteqdot (\lambda x\tau(b), x) \underset{3,5}{\equiv} \tau(b)$

$\tau(1\ b) \doteqdot (\lambda x\tau(1),\ \tau(b\ )\ \doteqdot (\lambda xx, \tau(b)) \underset{5}{\equiv} \tau(b)$

DA2. $\tau(d(cb)) \doteqdot (\lambda x\tau(d),\ \tau(cb)) \doteqdot (\lambda x\tau(d),\ (\lambda y\tau(c),\ \tau(b))) \underset{5}{\equiv}$

$\tau(d)_x[\tau(c)_y[\tau(b)]] \underset{3}{\equiv} \tau(d)_x[\tau(c)])_y[\tau(b)] \underset{5}{\equiv} \tau((dc)b)$

DA3. $\tau(a) \underset{4}{\equiv} \mathbf{I}$ for $a : A \to I$ since $\tau(a) \in I$

DA4. $\tau(\ell\langle a,b\rangle) \doteqdot (\lambda x\tau(\ell),\ \langle\tau(a),\ \tau(b)\rangle) \doteqdot$
$(\lambda x(\ell x),\ \langle\tau(a),\ \tau(b)\rangle) \underset{5}{\equiv} \ell\langle\tau(a),\ \tau(b)\rangle \underset{6}{\equiv} \tau(a).$

Similarly $\tau(\mathbf{r}\langle a,b\rangle) \equiv \tau(b)$.

DA5. $\tau(\langle\ell g,\ \mathbf{r}g\rangle) \doteqdot \tau(\ell g),\ \tau(\mathbf{r}g)\rangle \doteqdot \langle\ell\tau(g),\ \mathbf{r}\tau(g)\rangle \underset{2}{\equiv} \tau(g)$

DA6. $\tau(\varepsilon\langle c^{+}\ \ell,\mathbf{r}\rangle) \doteqdot (\lambda x\tau(\varepsilon),\ \langle\tau(c^{+}\ \ell),\ \tau(\mathbf{r})\rangle) \doteqdot$

$\doteqdot (\lambda x(\ell x,\ \mathbf{r}x),\ \langle\tau(c^{+}\ \ell),\ \tau(\mathbf{r})\rangle)$

$\underset{5,6}{\equiv} (\tau(c^{+}\ \ell),\ \tau(\mathbf{r})) \doteqdot ((\lambda x^A\tau(c^{+}),\ \tau(\ell)),\ \mathbf{r}x^{A\&B}).$

$\doteqdot ((\lambda x^A\lambda x^B\ (\lambda x^{A\&B}\tau(c),\ \langle x^A, x^B\rangle),\ \ell x^{A\&B}),\ \ \mathbf{r}x^{A\&B})$

$\underset{6}{\equiv} \lambda x^{A\&B}\tau(c),\ \langle\ell x^{A\&B},\ \mathbf{r}x^{A\&B}\rangle \underset{2}{\equiv} (\lambda x^{A\&B}\tau(c)\ x^{A\&B}) \underset{6}{\equiv} \tau(c).$

DA7. $\tau((\varepsilon\langle f\ell,\mathbf{r}\rangle)^{+}) \doteqdot \lambda x^{B}\,(\lambda x^{A\&B}\tau(\varepsilon\langle f\ell,\mathbf{r}\rangle),\ \langle x^{A},x^{B}\rangle);$

$\tau(\varepsilon\langle f\ell,\mathbf{r}\rangle) \doteqdot (\lambda x^{(B\supset C)\&B}\,\tau(\varepsilon),\ \tau(\langle f\ell,\mathbf{r}\rangle))$

$\doteqdot (\lambda x^{(B\supset C)\&B}\,(\ell x^{(B\supset C)\&B},\ \mathbf{r}x^{(B\supset C)\&B}),\ \langle \tau(f\ell)\ \tau(\mathbf{r})\rangle)$

$\underset{5,6}{\equiv} (\tau(f\ell),\ \tau(\mathbf{r})) \doteqdot ((\lambda x^{A}\tau(f),\ \ell x^{A\&B}),\ \mathbf{r}x^{A\&B})$

Hence $\tau((\varepsilon\langle f\ell,\mathbf{r}\rangle))^{+} \equiv \lambda x^{B}(\lambda x^{A\&B}\ ((\lambda x^{A}\tau(f),\ \ell x^{A\&B}),\ \mathbf{r}x^{A\&B}),\ \langle x^{A},x^{B}\rangle) \underset{5,6}{\equiv} \lambda x^{B}(\lambda x^{A}\tau(f),x^{A}),\ x^{B}) \underset{1}{\equiv} \lambda x^{B}(\tau(f),\ x^{B}\rangle \underset{1}{\equiv} \tau(f).$

The rules for deriving new relations $a \equiv b$ obviously preserve τ. The theorem is proved.

The next step is the analog of lemma 4.3. for $\mathbf{c}$, that is stability of $\mathbf{c}$ relative to $\equiv$ for terms. The most difficult part is checking relation (5) in lemma 4.5. below. First we prove a lemma concerning (what is called in proof theory) permutation and thinning rules.

LEMMA 4.4. If Γ,Γ' are lists of variables containing no repetitions, all variables from Γ are in Γ' and all free variables of t are in Γ, then

$$\mathbf{c}_{\Gamma'}\ (t) \equiv \mathbf{c}_{\Gamma}\ (t)\ \alpha_{x^{I}{}_{\Gamma'},x^{I}{}_{\Gamma}}$$

The proof is by induction on (the construction of) t.

1. $t \doteqdot x^{A}$ Then

$$\mathbf{c}_{\Gamma'}\ (t) \doteqdot \alpha_{x^{I}{}_{\Pi'},x^{A}{}_{\Sigma'},x^{A}};\ \mathbf{c}_{\Gamma}\ (t) \doteqdot \alpha_{x^{I}{}_{\Pi'},x^{A}{}_{\Sigma}x^{A}}$$

Use theorem 2.2.

2,3. $t \doteqdot \{b,a\}$ where $\{\}$ is $()$ or $\langle\ \rangle$. Use lemma 4.1.(i).

4. $t \doteqdot \lambda x^{A}b.$

$$\mathbf{c}_{\Gamma'}\ (t) \doteqdot \mathbf{c}^{+}_{\Gamma'x^{A}}(b) \equiv (\ \mathbf{c}_{\Gamma x^{A}}(b)\ \ \alpha_{x^{I}{}_{\Gamma'x^{A}},x^{I}{}_{\Gamma}\ x^{A}})^{+}$$

$$\underset{(5,\S3)}{\equiv}(\mathbf{c}_{\Gamma'x^{A}}(b)\ \langle\alpha_{x^{I}{}_{\Gamma'},x^{I}{}_{\Gamma}}\ell,\mathbf{r}\rangle)^{+} \underset{(iii)}{\equiv} \mathbf{c}^{+}\ (b)\ \alpha_{x^{I}{}_{\Gamma'},x^{I}{}_{\Gamma}}$$

5. $t \doteqdot \mathbf{f}b$ where $f \doteqdot r,\ l.$

$$\mathbf{c}_{\Gamma'}\ (t) \doteqdot \mathbf{f}\mathbf{c}_{\Gamma'}(b) \equiv \mathbf{f}\mathbf{c}_{\Gamma}(b)\ \alpha \doteqdot \mathbf{c}_{\Gamma}(t)\ \alpha$$

6. $t \doteqdot \mathbf{I}$ use DA3.

The lemma is proved.

LEMMA 4.5. $\mathbf{c}_\Gamma((\lambda x^A\ t^B,\ a^A)) \equiv \mathbf{c}_\Gamma(t^B_{x^A}[a])$.

Proof. Note that

$$\mathbf{c}_\Gamma((\lambda xt,a)) \doteq \varepsilon\ \langle \mathbf{c}^+_{\Gamma x}(t),\ \mathbf{c}_\Gamma(a)\rangle \underset{\text{(ii)}}{\equiv} \mathbf{c}_{\Gamma x}(t)\ \langle 1,\ \mathbf{c}_\Gamma(a)\rangle,$$

so it is sufficient to prove $\mathbf{c}_{\Gamma x}\langle 1,\ \mathbf{c}_\Gamma\ (a)\rangle \equiv \mathbf{c}_{\Gamma x}(t[a])$, or, using the abbreviation $\alpha' \doteq \langle 1,\ \mathbf{c}_\Gamma(a)\rangle$,

$$\mathbf{c}_{\Gamma x}(t)a' \equiv \mathbf{c}_\Gamma(t[a]) \qquad (7)$$

Assume first that x is not free in t. Then $t[a] \doteq t$ and (by lemma 4.4.) $\mathbf{c}_{\Gamma x}(t) \equiv \mathbf{c}_{\Gamma x}(t)\alpha_{\Gamma x,\Gamma} \equiv \mathbf{c}_\Gamma(t)\ell$ hence

$$\mathbf{c}_{\Gamma x}(t)a' \equiv \mathbf{c}_\Gamma(t)\ell\ \langle 1,\ \mathbf{c}_\Gamma\ (a)\rangle \equiv \mathbf{c}_\Gamma(t) \doteq \mathbf{c}_\Gamma(t[a])$$

as required.

So we assume below that x occurs free in t. We shall use induction on the construction of t.

1. $t \doteq x^A$; $t[a] \doteq a$; $\mathbf{c}_{\Gamma x}(t) \equiv \mathbf{r}$;

$$\mathbf{c}_{\Gamma x}(t)a' \equiv \mathbf{r}\ \langle 1,\ \mathbf{c}_\Gamma\ (a)\rangle \equiv \mathbf{c}_\Gamma(a) \doteq \mathbf{c}_\Gamma(t[a]).$$

2,3 $t \doteq \{b,e\}$, where $\{\}$ is () or $\langle\rangle$. Now $t[a] \doteq \{b[a],e[a]\}$;

$$\mathbf{c}_{\Gamma x}(t)a' \doteq \varepsilon^{\{\}}\ \langle \mathbf{c}_{\Gamma x}(b),\ \mathbf{c}_{\Gamma x}(e)\rangle\ a' \equiv \varepsilon^{\{\}}\ \langle \mathbf{c}_{\Gamma x}(b)\ a',\ \ \mathbf{c}_{\Gamma x}(e)a'\rangle$$

$$\equiv\ \varepsilon^{\{\}}\ \langle b[a],e[a]\rangle \doteq \mathbf{c}(t[a])$$

where $\varepsilon^{\{\}}$ is ε in ()-case and is empty in $\langle\rangle$-case.

4. $t \doteq \lambda y^D\ b$; $t[a] \doteq \lambda y^D\ b[a]$; $\mathbf{c}_\Gamma(t[a]) \doteq \mathbf{c}^+_{\Gamma y}(b[a])$

$$\mathbf{c}_{\Gamma x}(t)a' \doteq \mathbf{c}^+_{\Gamma xy}(b)a' \underset{\text{(iii)}}{\equiv} (\mathbf{c}_{\Gamma xy}(b)\ \langle a'\ \ell,\mathbf{r}\rangle)^+$$

$$(\mathbf{c}_{\Gamma xy}(b)\ \langle\langle \ell\ell,\mathbf{r}\rangle,\mathbf{r}\mathrm{L}\rangle\ \langle a'\ \ell,\mathbf{r}\rangle)^+$$

since we have $\alpha_{x'\Gamma xy,\ x'\ \Gamma yx} \equiv \langle\langle \ell\ell,\mathbf{r}\rangle,\mathbf{r}\ \ell\rangle$ by lemma 2.2.

We proceed:

$$\mathbf{c}_{\Gamma yx}(b)\ \langle\langle \ell\ell,\mathbf{r}\rangle,\mathbf{r}\ell\rangle\ \langle a'\ \ell,\mathbf{r}\rangle \underset{(\mathrm{i})}{\equiv}$$
$$\mathbf{c}_{\Gamma yx}(b)\ \langle\langle \ell\ell,\mathbf{r}\rangle\langle a'\ell,\mathbf{r}\rangle,\mathbf{r}\ell\ \langle a'\ \ell,\mathbf{r}\rangle\rangle \underset{\mathrm{DA4}}{\equiv}\ \mathbf{c}_{\Gamma yx}(b)\ \langle \ell a'\ell,\mathbf{r}\rangle,\ \mathbf{r}a'\ell$$
$$\underset{\mathrm{DA4}}{\equiv}\ \mathbf{c}_{\Gamma yx}(b)\ \langle\langle \ell,\mathbf{r}\rangle,\ \mathbf{c}_{\Gamma}(a)\ell\rangle \underset{\mathrm{DA5}}{\equiv} \mathbf{c}_{\Gamma yx}(b)\ \langle 1,\ \mathbf{c}_{\Gamma}(a)\ell\rangle$$
$$\equiv \mathbf{c}_{\Gamma yx}(b)\ \langle 1,\ \mathbf{c}_{\Gamma y}(a)\rangle \equiv \mathbf{c}_{\Gamma y}(b[a])$$

by induction hypothesis.

5. $t \doteq \mathbf{f}\ b.\ \mathbf{c}_{\Gamma x}(t)a' \doteq \mathbf{f}\ \mathbf{c}_{\Gamma x}(b)a' \equiv \mathbf{f}\ \mathbf{c}_{\Gamma}(b[a]) \doteq \mathbf{c}_{\Gamma}(t[a])$.

6. $t \doteq \mathbf{I}$. Then x^A does not occur in t.

The lemma is proved.

LEMMA 4.6. $t \equiv t'$ implies $\mathbf{c}_{\Gamma}(t) \equiv \mathbf{c}_{\Gamma}(t')$.

Proof. It is sufficient to prove that $\mathbf{c}$ preserves relations (1) - (6).

1. $\mathbf{c}_{\Gamma}(\lambda x(b,x)) \doteq \mathbf{c}^{+}_{\Gamma x}((b,x)) \doteq (\varepsilon\langle \mathbf{c}_{\Gamma x}((b)\mathbf{l}\rangle)^{+} \equiv (\varepsilon\langle\ \mathbf{c}_{\Gamma}(b)\mathbf{r},\mathbf{l}\rangle)^{+}$

by lemma 4.3. since $\alpha_{x^{I}\Gamma x^{A},x^{I}\Gamma} \equiv \mathbf{r}$. Use DA7.

2. $\mathbf{c}(\langle \ell d,\ \mathbf{r}\mathrm{d}\rangle) \doteq \langle \ell\mathbf{c}(d),\ \mathbf{r}\mathbf{c}(d) \underset{\mathrm{DA5}}{\equiv} \mathbf{c}(d)$
3. $\mathbf{c}(t) \doteq \mathbf{c}(t')$ if t, t' are congruent.
4. $\mathbf{c}(a^{I}) \underset{(\mathrm{DA3})}{\equiv} O \doteq \mathbf{c}(\mathbf{I})$.
5. $\mathbf{c}((\lambda xb,a) \equiv c(b[a])$ by lemma 4.5.
6. $\mathbf{c}(\mathbf{f}\langle a_l,a_r\rangle) \doteq \mathbf{f}\ \langle \mathbf{c}(a_l),\ \mathbf{c}(a_r)\rangle \equiv \mathbf{c}(a_f)$ where $f \doteq l,r$

The lemma is proved.

THEOREM 4.7. For any combinators a,b

$$a \equiv b \qquad \text{iff} \qquad \tau(a) \equiv \tau(b).$$

Proof. If $a \equiv b$ then $\tau(a) \equiv \tau(b)$. by lemma 4.3. If $a,b : A \to B$ and $\tau(a) \equiv \tau(b)$ then $\mathbf{c}(\tau(a)) \equiv \mathbf{c}(\tau(b))$ by lemma 4.6, so

$$a\ \mathbf{r} \equiv\ b\ \mathbf{r} : I\&A \to B$$

by lemma 4.2. Hence $a \underset{\mathrm{DA4}}{\equiv} a\ \mathbf{r}\ \langle O_A,\ 1_A\rangle \equiv b\mathbf{r}\ \langle O_A,\ 1_A\rangle \underset{\mathrm{DA4}}{\equiv} b$ which had to be proved.

5. Conversion, reduction and normal forms

Let R be some binary relation between elements of some set (say of terms). An R-reduction sequence is a sequence (finite or infinite)

$$t_0,\ t_1,\ \ldots \tag{1}$$

satisfying $t_i\ R\ t_{i+1}$ for all i. The sequence (1) is an R-reduction of t to t' if (1) is finite, $t_0 \equiv t$ and the last term of (1) is t'. The notation $t \models_R t'$ is read «t R-reduces to t'» and means that there is an R-reduction of t to t'. The notation $t \models^{n}_{R} t'$ is read «t R-reduces to t' in n steps» and means there is an R-reduction of length n. $t \vdash t'$ means (t R t'. or $t \equiv t'$). A term is R-normal (or is in R-normal form) if it R-reduces to no term except itself. Let us assume R fixed and drop references to R whenever possible. We shall describe one way of establishing the uniqueness of the normal forms modulo some equivalence relation (cf. [14]).

Let $\sim$ be some equivalence relation on the domain of R.

THEOREM 5.1. Assume that ((1) - (3)):

(1) normality is stable under $\sim$, i.e. $t \sim s$ for normal t implies normality of s;

(2) r is well-founded, i.e. all R-reduction sequences are finite;

(3) for any t_0, t_1, s_0, s_1 such that $s_0 \dashv t_0 \sim t_1 \vdash s_1$ one can find s'_0, s'_1 for the following diagram:

$$\begin{array}{ccccc} t_0 & \sim & t_1 & \vdash & s_1 \\ \top & & & & ⫪ \\ s_0 & \vdash & s'_0 & \sim & s'_1 \end{array} \tag{2}$$

Then the normal form always exists and is unique up to equivalence, that is

(4) If $t_0 \sim t_1$, $t_i \vdash \bar{t}_i$ and $\bar{t}_i$ are normal ($i = 0, 1$) then $\bar{t}_0 \sim \bar{t}_1$.

Condition (3) can be expressed as follows: if $t \vdash s_0$ and $t \vdash s_1$, then one can find s' such that the diagram

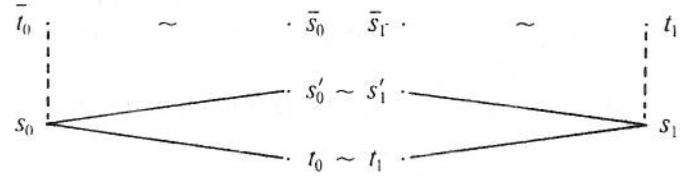
$$\begin{array}{ccc} t & \vdash & s_1 \\ \top & & ⫪ \\ s_0 & \vdash & s' \end{array}$$

commutes up to congruence.

Proof. Let the reduction tree $\mathfrak{T}_t$ for a term t be constructed as follows: t itself is assigned to the bottom (lowermost) node of $\mathfrak{T}_t$. If the term s is assigned to the node a then all terms s' such that sRs' are assigned to immediate predecessors of a. Hence paths in $\mathfrak{T}_t$ are exactly reduction sequences of t and the terms assigned to the uppermost nodes of $\mathfrak{T}_t$ are in normal form.

In view of the clause (2) of the theorem we can use induction on (the disjoint sum of) $\mathfrak{T}_{t_0}$ and $\mathfrak{T}_{t_1}$. If one of t_0, t_1 is normal, we have (4) by the clause (1) of the theorem. This proves the induction base, that is (4) for uppermost nodes.

Let $t_i \vdash s_i \vdash \bar{t}_i$, $i = 0, 1$. By (3) we find s'_0, s'_1 and by the induction hypothesis we have $\bar{s}_0 \sim, \bar{s}_1$ for the normal forms $\bar{s}_i$ of terms s'_i, $i = 0, 1$. Applying the induction hypothesis to pairs (s_i, s) we have $\bar{t}_i \sim \bar{s}_i$ $(i = 0, 1)$.



Transitivity of ~ implies $\bar{t}_0 \sim \bar{t}_1$ which was to be proved.

We shall apply this theorem in the situation where the relation r is defined via some relation $\mapsto$ between terms: tRt' means that t' is obtained from t by replacing a subterm c of t by c', where $c \mapsto c'$ and some proviso for t, t', c, c' is satisfied. In this case both the relation $c \to c'$ (and sometimes tRt') and the passage from c to c' are called *conversions*, and c is said to *convert* to c'.

Another strategy of proof is used when the well-foundedness proof is too complicated or R is not well-founded (for example there are terms without a normal form). Then $\models$ is replaced by $\vdash$ in diagram (2), for some relation R^+ equivalent to the given R.

Let CR (from Church-Rosser) be the following property: for any t', t'', s', s'' such that $s' \dashv t' \sim t \vdash s''$ there are $\tilde{s}$, $\tilde{s}'$ such that $s' \vdash \tilde{s}' \sim \tilde{s}'' \dashv s''$. In other words: for given $s' \dashv t \vdash s''$ one can find $\tilde{s}$ such that

$$\begin{array}{ccc} t & \vdash & s' \\ \top & & \top \\ s'' & \vdash & s \end{array} \qquad (3)$$

commutes up to congruence.

THEOREM 5.2. (Church-Rosser property). Let the relation R^+ satisfy CR and $t \models_R t'$ be equivalent to $t \models_{R+} t'$ for any t, t'. Then an R-normal form is unique up to congruence.

Proof. R-normality is the same as R^+-normality by the assumption on R, R^+, so $R = R^+$ can be assumed. Let us prove by induction on $m + n$ that diagram (3) can be generalized modulo congruence: from $t \overset{m}{\models} s'$ and $t \overset{n}{\models} s''$ it follows that $s' \overset{n}{\models} \tilde{s}$ and $s'' \overset{m}{\models} \tilde{s}$ for some $\tilde{s}$.

$$\begin{array}{ccc} t & \overset{m}{\models} & s' \\ {}_n\top\!\!\top & & \top\!\!\top_n \\ s'' & \overset{m}{\models} & \tilde{s} \end{array} \quad \text{or more precisely} \quad \begin{array}{ccccc} t'' & \sim & t' & \overset{m}{\models} & s' \\ {}_n\top\!\!\top & & & & \top\!\!\top_n \\ s'' & & \overset{m}{\models} & & \tilde{s}'' \sim \tilde{s}' \end{array} \qquad (4)$$

Let us argue modulo congruence. If $m = 0$ put $s' \sim s''$; similarly for $n = 0$. If $m, n \geq 1$, use (3) and the inductive assumption (twice):

$$\begin{array}{ccccc} t & \vdash & t_1 & \overset{m-1}{\models} & s' \\ \top & & \top & & \top \\ t''_1 & \vdash & \tilde{t}_1 & \overset{m-1}{\models} & s'_1 \\ {}_{n-1}\top\!\!\top & & {}_{n-1}\top\!\!\top & & \top\!\!\top_{n-1} \\ s'' & \vdash & s''_1 & \overset{m-1}{\models} & \tilde{s} \end{array}$$

which proves (4). If s' and s'' in (4) are normal then $s' \sim s'' \sim \tilde{s}$ which was to be proved.

6. Contraction and (λ, $\langle\rangle$)-reductions

We consider two reduction relations and prove the existence and uniqueness of normal forms for each of them. In §7 a similar result for their combination will be established and used in a decision algorithm for $a \equiv b$.

The relation R_1 will be defined by conversions corresponding to (1), (2) in §4. Put

$$b^+ = \begin{cases} \lambda x^A\ (b, x^A), \text{ if } b \in (A \supset B),\ x^A \text{ is not free in } b \\ \langle \ell b,\ r\mathbf{b} \rangle, \text{ if } b \in \ (A \,\&B) \end{cases}$$

and consider conversions

$$b^+ \mapsto b \qquad (K1)$$

These conversions are called contractions of b^+ into b. They are called η-conversions in the literature on λ-calculus. Let R_1 be the relation defined by these conversions: tR_1t' means that $t \sim s_y\ \{b^+\}$ and $t' \sim s_y\{b\}$ that is t' is obtained by replacing some occurrence of b^+ in t by b. (Recall that $s_y\{b\}$ is the result of substituting b for free occurrences of y in b without renaming of variables.)

LEMMA 6.1. The relation $R1$ is well-founded.

Proof. Let $\rho_1\,(t)$ be the length of the term t. Since $\rho_1\,(t^+) > \rho_1\,(t)$, it is obvious that $\rho_1\,(t)$ strictly decreases when conversion is applied

$$tR_1\,t' \qquad \text{implies} \qquad \rho_1\,(t) > \rho_1\,(t'),$$

This implies well-foundedness, since the length of any $R1$-reduction of t is less than the length of t. □

Let us check condition (3) of theorem 5.1.

LEMMA 6.2. For R_1-conversion the diagram

$$\begin{array}{ccc} t & \vdash & s_1 \\ \top & & \top \\ S_0 & \vdash & \tilde{s} \end{array} \tag{3}$$

with given t, s_0, s_1 can be completed by suitable $\tilde{s}$ (modulo congruence).

Proof. If one of the conversions in (1) is simply the identity or $s_1 \sim s_0$, everything is obvious. If converted occurrences are non-overlapping, that is $t \sim u[c_0^+,\ c_1^+]$ with $s_0 \sim u[c_0,\ c_1^+]$ and $s_1 \sim u[c_0^+,\ c_1]$, put $\tilde{s} \sim u[c_0,\ c_1]$. So assume one of the converted occurrences (say one converted in passing to s_1) is inside another, which we shall assume to be the outermost one to simplify notation. Under these assumption we have: $t \mp u^+[c^+] \mapsto u[c^+] \mp s_0$; $s_1 \sim u^+[c]$. Put $s_1 \sim u[c]$ which concludes the proof.

THEOREM 6.3. Any term has a unique $R1$-normal form (modulo congruence).

It follows from lemmas 6.1, 6.2 and the theorem 5.1.

Consider now conversions corresponding to the remaining relations (4) - (6) in §4. Put

$$t^- \mp \begin{cases} \mathbf{I} & \text{if } t \in I \\ b_x[a] & \text{if } t = (\lambda x b, a) \\ a_f & \text{if } t = \mathbf{f}\langle a_l,\ a_r\rangle,\ \ f = l,\ r \end{cases}$$

Consider conversions

$$t \vdash t^- \tag{K2}$$

and let R_2 be the relation defined by these conversions: uR_2u' means $u \sim v_x\{t\}$, $u' \sim v_x\{t^-\}$, for some v, t that is u' is obtained from u by replacing some occurrence of t by t^-.

LEMMA 6.4. Any term has an $R2$-normal form.

Proof. We use the term *cuts* for the left hand sides t' of conversions (4) - (6). The degree of the conversion $t \mapsto I$ is 0 by definition; the degree of $(\lambda x^A b^B, a) \mapsto b[a]$ and $\mathrm{f}\langle a_l^A, a_r^B\rangle \mapsto a_f$ is by definition $d(A) + d(B) + 1$. *Elimination of a* given *cut* is its replacement by the right hand side of the corresponding conversion. Let the cut-degree $\rho_2(t)$ of a term t be the maximum degree of cuts in t. We apply induction on $\rho_2(t)$.

Induction base. Only conversions $t \mapsto I$ are applicable. Apply all of them and obtain a normal term.

To prove the induction step on $\rho_2(t)$ we use induction on the number k of cuts of degree $\rho_2(t)$. Choose one of the innermost cuts of degree $\rho_2(t)$ and eliminate it. As a result some subterm c of t is replaced by c^- and t is transformed into t'.

Let us prove that t' contains no cuts of degree $> \rho_2(t)$ and k is decreased by 1. Suppose the conversion $\ell\langle a^A, b^B\rangle \mapsto a^A$ was applied. Then to every occurrence V' of a subterm e' in t' corresponds a unique occurrence V in t of the term e of the same type as e', and e' either is e or is obtained from e by replacing $\ell\langle a, b\rangle$ by a. Now if V' is a cut and V is not, then V' is an occurrence of the term $\mathrm{f}a$ or (a,c). Then the degree of V' is $d(A) < d(A) + d(B) + 1$. So no new cut of degree $\geq \rho_2(t)$ arises and one cut of this degree was eliminated.

Assume that the passage $t \vdash t'$ was according to $(\lambda x^A \mathrm{b}^B, a) \mapsto b[a]$. Then new (compared to ones in t) cuts in t' are cuts of the form (a^A, d') or $\mathrm{f}a^A$ which appeared instead of subterms (x^A, d) or $\mathrm{f}x^A$. Their degree is again $d(A) < d(A) + d(B) + 1$. The theorem is proved.

To prove the uniqueness of $R2$-normal form we apply theorem 5.1 and with that aim define inductively the relation $\Rightarrow$ corresponding to R using a device due to Tait (cf.[14]). The meaning of $t \Rightarrow t'$ is approximately «t' is obtained from t by several conversions in nonoverlapping occurrences». The defining rules for $\Rightarrow$ have the property: the left-hand sides of premisses (relations above the line) are simpler than those of the conclusion, except for the congruence rule.

$0.^{\Rightarrow}\ t \Rightarrow t \qquad 1.^{\Rightarrow}\ t^I \Rightarrow \mathbf{I} \qquad 2.^{\Rightarrow}\ \dfrac{a \Rightarrow b}{a \Rightarrow \tilde{b}}\ \tilde{b}$ is congruent to b

$3.^{\Rightarrow}\ \dfrac{b \Rightarrow b' \quad a \Rightarrow a'}{(\lambda x b, a) \Rightarrow b'_x[a']} \quad 4.^{\Rightarrow}\ \dfrac{a \Rightarrow a'}{\ell\langle a, b\rangle \Rightarrow a'} \quad 4.^{\Rightarrow}\ \dfrac{b \Rightarrow b'}{\mathbf{r}\langle a, b\rangle \Rightarrow b'}$

$5.^{\Rightarrow}\ \dfrac{a \Rightarrow a'}{f\ a \Rightarrow f\ a'}\ f \doteq \ell, \mathbf{r}, \lambda x \quad 6.^{\Rightarrow}\ \dfrac{a \Rightarrow a' \quad b \Rightarrow b'}{\{a,b\} \Rightarrow \{a', b'\}} \quad \{\} \doteq (), \langle\rangle$

A derivation of $t \Rightarrow t'$ according to $0^{\Rightarrow}$-$6^{\Rightarrow}$ is called a *demonstration* of $t \Rightarrow t'$.

Proof. Any conversion according to R_2 is simulated by the correspon-ding rule $1^{\Rightarrow}$-$4^{\Rightarrow}$, and replacement inside terms is simulated by $5^{\Rightarrow}$, $6^{\Rightarrow}$.

A demonstration of $t \Rightarrow t'$ is step-by-step translated in a reduction $t \models_{r_1} t'$ (that is the latter is constructed by recursion on the former).

LEMMA 6.6. If $a \Rightarrow a'$, $b \Rightarrow b'$, then $b[a] \Rightarrow b'\,[a']$.

Proof. We use induction on the demonstration of $b \Rightarrow b'$. The induction base is established by induction on the construction of b: $b[a] \Rightarrow b[a']$ is derived from $a \Rightarrow a'$ by rules $5^{\Rightarrow}$, $6^{\Rightarrow}$. The proof of the induction step uses the stability of all the rules under substitution of (possibly different) terms in the left-hand and right-hand sides: if

$$\frac{b_1 \Rightarrow c_1 \;\ldots\; b_k \Rightarrow c_k}{d \Rightarrow e}$$

is an inference according to some rule, then

$$\frac{b_1[a] \Rightarrow c_1[a'] \;\ldots\; b_k[a] \Rightarrow c_k[a']}{d[a] \Rightarrow c[a']}$$

is again an inference according to the same rule.

LEMMA 6.7. If $t \Rightarrow t'$, $t \Rightarrow t''$ then there is a $\tilde{t}$ such that

$$\begin{array}{ccc} t & \Rightarrow & t' \\ \Downarrow & & \Downarrow \\ t'' & \Rightarrow & \tilde{t} \end{array}$$

Proof is by induction on the sum of the lenghts of the demonstrations for $t \Rightarrow t'$, $t \Rightarrow t''$. Let $t \Rightarrow t'$ be inferred by $i^{\Rightarrow}$ and $t \Rightarrow t''$ by $j^{\Rightarrow}$. By symmetry we assume $i \leq j$.

If $i = 0$ put $\tilde{t} \doteqdot t''$, if $i = 1$ put $\tilde{t} \doteqdot \mathbf{I}$.

If $i = 2$, apply the induction hypothesis, $2^{\Rightarrow}$ and $j^{\Rightarrow}$:

$$\begin{array}{l} \dfrac{t \Rightarrow a}{t \;\Rightarrow \tilde{a} \doteqdot t'} \quad \text{where } t'' \Rightarrow \tilde{t}^{\text{TM}} \Leftarrow a \\ j \Downarrow \qquad \Downarrow j \\ t'' \qquad \tilde{t} \end{array}$$

If $u = 3$ then $j = 3, 6$. Apply the induction hypothesis and lemma 6.6:

$$j = 3:\quad \begin{array}{ccc} (\lambda xb,\ a) & \Rightarrow & b'\ [a'] \\ \Downarrow & & \Downarrow \\ b''[a''] & \Rightarrow & \tilde{b}[\tilde{a}] \end{array} \qquad j = 6:\quad \begin{array}{ccc} (\lambda xb,\ a) & \Rightarrow & b'\ [a'] \\ \Downarrow & & \Downarrow \\ (\lambda xb'',\ a'') & \Rightarrow & \tilde{b}[\tilde{a}] \end{array}$$

If $i = 4$ then $j = 4, 5$. For $j = 4$ we use the induction hypothesis:

$$\begin{array}{lcl} \ell\langle a,b\rangle & \Rightarrow & a' \\ & & \\ a' & \Rightarrow & \tilde{a} \end{array}$$

If $j = 5$ we have $t'' \doteq \ell s$ with $\langle a,b\rangle \Rightarrow s$, which implies $s \doteq \langle a'', b''\rangle$ with $a \Rightarrow a''$, $b \Rightarrow b''$ with shorter derivations by induction on the length of the demonstration. We have:

$$\begin{array}{ccc} \ell\langle a,b\rangle & \Rightarrow & a' \\ \Downarrow & & \Downarrow \\ \ell\langle a'',\ b''\rangle & \Rightarrow & \tilde{a} \end{array}$$

If $i = 5, 6$ then our assumption $i \leq j$ implies $i = j$, and we use the induction hypothesis. For example if $i = j = 6$ we have

$$\begin{array}{ccc} \{a,\ b\} & \Rightarrow & \{a',\ b'\} \\ \Downarrow & & \Downarrow \\ \{a'',\ b''\} & \Rightarrow & \{\tilde{a},\ \tilde{b}\} \end{array}$$

This concludes the proof.

THEOREM 6.8. Any term has an R_2-normal form which is unique up to congruence.

Proof. Existence follows from lemma 6.4 and uniqueness from theorem 5.2 and lemma 6.7.

7. Reduced normal forms and the decidability of ≡

Let $\bar{\bar{t}}$ denote the R_2-normal form of the term t. The *reduced normal form* $\bar{t}$ of the term t is by definition the R_1-normal form of the term $\bar{\bar{t}}$. By theorem 6.8 and 6.3 the reduced normal form exists and is unique up to congruence. We shall show that $a \equiv b$ is equivalent to $\overline{\tau(a)} \sim \overline{\tau(b)}$ for any combinators a, b. First we state the transformation rules for R_2-normal forms.

LEMMA 7.1. (i) $a \underset{R_2}{\models} b$ implies $\bar{\bar{a}} \sim \bar{\bar{b}}$;

(ii) $\overline{\overline{\lambda xa}} \sim \lambda x\bar{\bar{a}}$; $\overline{\overline{\langle a, b\rangle}} \sim \langle\bar{\bar{a}}, \bar{\bar{b}}\rangle$

(iii) $\overline{\overline{\mathbf{f}a}} \sim \mathbf{f}\bar{\bar{a}}$; $\overline{\overline{(a,b)}} \sim \overline{\overline{(\bar{\bar{a}},\bar{\bar{b}})}}$; $f \doteq l,r$

(iv) $\overline{\overline{\mathbf{f}a}} \sim f\bar{\bar{a}}$; $\overline{\overline{(a,b)}} \sim (\bar{\bar{a}},\bar{\bar{b}})$ for $\bar{\bar{a}} \neq \langle c,d\rangle$, λyc;

(v) $\overline{\overline{t\{a\}}} \sim \overline{\overline{\bar{\bar{t}}\{\bar{\bar{a}}\}}} \sim \overline{\overline{t\{\bar{\bar{a}}\}}}$; $\sim \overline{\overline{\bar{\bar{t}}\{a\}}}$;

(vi) $\overline{\overline{a^+}} \sim \begin{cases} \bar{\bar{a}} \text{ if } \bar{\bar{a}} \doteq \lambda xb, \langle b,c\rangle \\ \bar{\bar{a}}^+ \text{ otherwise.} \end{cases}$

Proof. (i) $a \vdash b \vdash \bar{\bar{b}}$, $a \vdash \bar{\bar{a}}$ implies $\bar{\bar{a}} \sim \bar{\bar{b}}$ by the uniquess of R_2-normal forms.

(ii) $a \vdash \bar{\bar{a}}$ implies $\lambda xa, \vdash \lambda x\bar{\bar{a}}$ and $\lambda x\bar{\bar{a}}$ is in R_2-normal form. By uniqueness $\lambda xa \sim \lambda x\bar{\bar{a}}$. For $\langle a, b\rangle$ and (iii) the proof is similar. The argument is the same for (iv): both $\mathbf{f}\bar{\bar{a}}$ and $(\bar{\bar{a}},\bar{\bar{b}})$ are in R_2-normal form if the proviso is satisfied.

(v) is proved by induction on (the construction of) t. The base is obvious and the induction step follows from (ii), (iii).

Obviously $a^+ \vdash \bar{\bar{a}}^+$. If $\bar{\bar{a}} \neq \lambda xb$, $\langle b, c\rangle$, then $\bar{\bar{a}}^+$ is already in normal form If $\bar{\bar{a}} \doteq \lambda xb$, then $\bar{\bar{a}}^+ \sim \lambda x\,(\lambda xb, x) \vdash \lambda xb \doteq \bar{\bar{a}}$. Similarly if $\bar{\bar{a}} \doteq \langle b, c\rangle$ then $\bar{\bar{a}}^+ \doteq \langle \ell\langle b, c\rangle, \mathbf{r}\langle b, c\rangle\rangle \vdash \langle b, c\rangle \doteq \bar{\bar{a}}$.

LEMMA 7.2. $\overline{\overline{t_x\{a^+\}}} \underset{R_1}{\models} \overline{\overline{t_x\{a\}}}$ up to congruence.

Proof. By lemma 7.1 (v) we assume t to be normal, and by lemma 7.1 (iv) we assume $\bar{a} \neq \lambda xb$, $\langle b, c\rangle$. Consider two classes of occurrences of x in t: the first class consist of all occurrences generated by subterms of the form $\mathbf{f}x$, (x, v), and the second class consists of all remaining occurrences of x. Writing t in the form $u_{z,w}\,\{x, x\}$, where z is substituted for all occurrences of the first class and w for all occurrences of the second class, we can assume that there are no occurrences of the first class: $\overline{\overline{(a^+, v)}} \sim \overline{\overline{(a, v)}}$; $\overline{\overline{\mathbf{f}a^+}} \sim \overline{\overline{\mathbf{f}a}}$. But then the terms $t\{\bar{\bar{a}}^+\}$ and $t\{\bar{\bar{a}}\}$ are both in R_2-normal form, so that by 7.1 (i) we have $\overline{\overline{t\{a\}}} \sim t\{\bar{\bar{a}}\}$, $\overline{\overline{t\{a^+\}}} \sim \overline{\overline{t\{\bar{\bar{a}}^+\}}}$. Now it is sufficient to note that under our assumptions concerning occurrences of x we have $t\{\bar{\bar{a}}\} \underset{R_1}{\models} t\{\bar{\bar{a}}^+\}$.

THEOREM 7.3. $t \equiv s$ is equivalent to $\bar{t} \sim \bar{s}$ for any terms t, s.

Proof. Let $\bar{t} \sim \bar{s}$. All reductions are instances of the relations (1) - (6) of §3, so $t \equiv \bar{t}$, $s \equiv \bar{s}$, hence $t \equiv s$.

To prove the converse implication it is sufficient to show that $\overline{t\{a\}} \sim t\{a^\sigma\}$, where $\sigma \mp +, -$ and apply induction on the length of the demonstration of $t \equiv s$.

We have $\overline{t\{a^-\}} \sim \overline{t\{\bar{\bar{a}}^-\}} \sim \overline{t\{\bar{\bar{a}}\}} \sim \overline{t\{a\}}$ by lemma 7.1 (i) (v). Hence $\overline{t\{a^-\}} \sim t\{a\}$.

By lemma 7.2. and the uniqueness of R_1-normal forms we have $\overline{t\{c^+\}} \sim t\{c\}$ which concludes the proof.

THEOREM 7.4. $a \equiv b$ is equivalent to $\overline{t\{a\}} \sim \overline{t\{b\}}$ for any combinators $a, b : A \rightarrow B$.

This follows from theorems 4.7 and 7.3.

Note 1. Theorem 7.3 shows that the reduced normal form $\bar{t}$ of the term is also its (unique) normal form relative to conversion (1) - (6), §6 applied in arbitrary order, and not only in the order described at the beginning of §6. In fact if $t \vdash s$ by (1) - (6) then $t \equiv s$ and so $\bar{t} \sim \bar{s}$ by theorem 7.3. This method of proving uniqueness of normal forms is due to Hanatani [16].

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